A Compound Poisson-Gamma Trade Model

Thomas Baranga*
IR/PS, UCSD
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Abstract

The paper builds a new statistical model of aggregate trade flows from microfoundations, which can be estimated by maximum likelihood, and is not subject to three important potential biases identified in the literature on gravity equation estimation: sample selection, heteroskedasticity, and heterogeneity bias. The model delivers estimates of the determinants of the intensive and extensive margins of exporting, as well as the number of local and exporting firms, using only aggregate trade flows. The model is tested on simulated data and compares favourably to other widely used estimation techniques.

1 Introduction

This paper develops a new empirical model to estimate the significance of trade costs on bilateral trade flows. It builds on the theoretical literature that has developed general equilibrium models of trade among firms with heterogeneous productivity. Central to papers in this literature, such as Eaton & Kortum (2002), Helpman, Melitz & Rubinstein (2008) and Eaton, Kortum & Sotelo (2012), is the idea that firms draw their productivity from a distribution. The variance of this distribution is critical for generating theoretical variance in trade flows, and papers such as Helpman et al. (2008) and Eaton et al. (2012) exploit it in their estimation methodology. However, typically estimation strategies also rely on atheoretical variation to empirically estimate these models. The main contribution of this paper is to develop a model in which the theoretical variation directly provides all of the empirical identification.

*Baranga: IR/PS 0519, University of California - San Diego, 9500 Gilman Drive, La Jolla, CA 92093 (email: tbaranga@ucsd.edu; 858 822 2877).
The paper is also related to a literature on appropriate estimation techniques for the popular gravity equation of bilateral trade, which is vulnerable to several known biases: selection bias, heterogeneity bias, and heteroskedasticity bias. A range of techniques have been proposed to address subsets of these, but existing methods cannot be combined to control for all three simultaneously. The empirical model developed in this paper is robust to all three biases.

Bilateral trade datasets are widely exploited in empirical research, in part because there are many more potential bilateral than country-specific observations. However, a significant proportion of bilateral trade flows between country pairs are zeros, which raises interesting econometric problems. It is very popular to log-linearise the gravity equation\(^1\), but as argued by Santos Silva & Tenreyro (2006) this may introduce several biases.

Dropping observations with no trade is a source of selection bias. In Monte Carlo simulations Santos Silva & Tenreyro (2006) show that adding 1 to all trade flows to include the log of true zeros in a regression leads to inaccurate estimates. In their Monte Carlo exercises a Tobit model, as proposed by Eaton & Tamura (1994), which predicts positive trade only if a latent variable exceeds a threshold, is also quite severely biased.

A major contribution of Helpman et al. (2008) was to develop an empirical method to estimate trade among heterogeneous firms while also controlling for the selection bias induced by log-linearising zero trade flows. They use a two-stage Heckman (1979) model to predict positive trade, which is more flexible than a Tobit model, and also elegantly corrects for the “heterogeneity” bias that they identify as arising from the aggregation of exports by firms with differing productivities, a potentially important issue for researchers using national, rather than firm-level, data.

Unfortunately, this does not address all of the known issues in estimating the gravity model. Santos Silva & Tenreyro (2006) show that log-linearisation techniques are biased if there is heteroskedasticity in the error term, as the expectation of the log of a random variable is not equal to the log of the expectation. Log-linearising heteroskedastic errors introduces correlation between the error and the independent variables, since the expected log depends on the variance of the error, which is a function of the independent variables when the errors are heteroskedastic. Santos Silva & Tenreyro (2006) argue that this bias is economically large in the setting of a standard gravity equation, and propose estimating the determinants of trade in levels using a Poisson pseudo-maximum-likelihood (PPML) estimator. In both their original and a follow-up paper (Santos Silva & Tenreyro (2014)) they strongly reject

\(^1\)e.g. Anderson & van Wincoop (2003)
that bilateral trade flows are homoskedastic, and this heteroskedasticity-bias could account for the divergence between traditional and PPML estimates.

Furthermore Santos Silva & Tenreyro (2014) show that the Helpman et al. (2008) model cannot be modified to account for heteroskedasticity. This leaves an unsatisfactory tension in the empirical literature, as none of the estimation techniques are robust against all the known biases (PPML does not address the question of heterogeneity bias).

This paper microfounds a compound Poisson-Gamma model of trade. Discreteness is introduced by allowing for a finite number of firms. Firms draw their productivity from a continuous distribution, and as in Melitz (2003), Helpman et al. (2008) and Eaton et al. (2012), face fixed costs of entering export markets which only relatively productive firms will choose to incur. The number of firms who choose to enter export markets follows a Poisson process. However, each exporting firm’s sales are a continuous variable that depend on their idiosyncratic productivity, and are distributed as an exponential with a mean distorted upwards by the selection effect. The exponential distribution has tractable aggregation properties, so that adding together the sales of a given number of exporters, aggregate trade follows a distorted gamma distribution. Finally, accounting for the randomness in the number of exporting firms yields a compound Poisson-Gamma for the unconditional distribution of aggregate bilateral trade.

The model can be estimated in levels, avoiding log-linearisation, and the variance of trade is heteroskedastic, consistent with the findings of Santos Silva and Tenreyro. The model can incorporate observations in which no firms export between a country-pair, so does not suffer from selection bias. The aggregation explicitly accounts for the varying productivities of participating firms, addressing HMR’s heterogeneity bias.

Eaton et al. (2012) is very close in spirit to this exercise. They also model a finite number of firms with heterogeneous productivity. One distinction is that rather than invoking the Poisson approximation to the Binomial distribution, they model the number of firms getting a productivity draw as directly following a Poisson distribution, and that the productivity is drawn from a Pareto distribution. This has the disadvantage that firm-level exports inherit a Pareto distribution, which renders the distribution of aggregate trade intractable. Their approach yields tractable expressions for mean trade shares, but not for trade levels, which may be of independent interest. Their model is augmented by statistical errors (multinomial logit and gamma) which are not microfounded in the theory but necessary for estimation of the model.

This paper offers a complementary approach to measuring the impact of trade frictions on trade flows, which may also find uses in other settings.
in which it is convenient to analyse the pooled outcome of many agents’ decisions with aggregate data.

2 Model

The model builds on a Melitz model (Melitz (2003)) of trade among firms with heterogeneous productivity. However, it departs from most implementations of the Melitz model (eg Helpman et al. (2008)) in assuming a Fréchet distribution for unit input requirements, rather than Pareto. The goal of the exercise is to derive a tractable closed-form distribution for aggregate bilateral trade flows. Firm-level trade flows inherit a Pareto distribution if productivity is assumed distributed Pareto. Unfortunately the distribution of the sum of Pareto random variables is a very complicated object, unlikely to yield easily to estimation.

Instead, we will assume that unit labour requirements are distributed Fréchet. This then implies that firm-level trade inherits a Weibull distribution. While in general the sum of Weibull’s is also very unwieldy in a special case of the underlying Fréchet the distribution of firm-sales collapses to an exponential, which is tractable.

The economic model is the Helpman et al. (2008) model adapted to a finite number of firms. The representative household in country i has a CES

\(^2\)For the derivation of sums of Pareto random variables, see Ramsay (2006) and Ramsay (2008)

\(^3\)The distribution of the sum of the log of Pareto random variables is easily computed; but unfortunately the sum of the log of firms’ trade is not an empirically useful object to model.

\(^4\)This may appear similar to Eaton & Kortum (2002), but there is an important difference. They assume that firm productivity, which is the inverse of the unit labour requirement, is distributed Fréchet. The assumption that unit labour follows a Fréchet distribution implies that firm productivity follows a Weibull distribution. The Fréchet is an extreme value distribution, which could represent the maximum of a series of draws from some other underlying distribution. A natural interpretation of productivity following a Fréchet arises in a setting in which firms are presented with a set of potential business opportunities and implement the most productive.

Modelling unit labour requirements as following a Fréchet instead could reflect an environment in which a firm is identified with a given line of business, which requires managing multiple input processes. A bottleneck in any of the input processes delays final production, so that overall costs reflect the cost of mastering the hardest process. This modelling assumption is motivated by the analytical tractability that it delivers.

\(^5\)Nadarajah (2008) reports that as of 2008 there were no known results for the distribution of the sum of Weibull random variables. Yilmaz & Alouini (2009) provide a distribution, although as in the case of Pareto random variables, it does not appear to be easily estimable.
utility function over consumption $x$ of goods varieties $l$

$$u_i = \left( \sum_{l \in B_i} x_i(l)^{\frac{1}{\epsilon}} dl \right)^{\frac{1}{\epsilon-1}}, \epsilon > 1$$

where $B_i$ is the set of available varieties in country $i$.

Given national income $Y_i$, demand for each variety $l$ is

$$x_i(l) = \frac{\tilde{p}_i(l)^{-\epsilon} Y_i}{P_i^{1-\epsilon}}$$

where $\tilde{p}_i(l)$ is the local (country $i$) price of variety $l$, and $P_i$ is country $i$’s ideal price index, given by

$$P_i = \left( \sum_{l \in B_i} \tilde{p}_i(l)^{1-\epsilon} dl \right)^{\frac{1}{1-\epsilon}}$$

Each country $j$ has $N_j$ firms, but not all firms will enter all markets. Firm $l$ in country $j$ requires $c_j a(l)$ inputs to produce one unit of output. The main departure from Helpman et al. (2008) and Eaton et al. (2012) is that rather than assuming that firm productivity is distributed Pareto, we assume that firm $l$’s unit input requirements $(a(l))$ are drawn from a Fréchet ($\kappa$) distribution

$$F(a \leq \alpha) = e^{-a^{-\kappa}}$$

In addition to the costs of producing for the local market, exporting firms also face both fixed and variable costs to serve a foreign market. The variable costs are of the “iceberg” form, so that an additional percentage, $\tau_{ij}$, must be shipped in order for one unit to be delivered, to accommodate the real resources dissipated in transit from $j$ to $i$. Irrespective of the sales volume, a firm exporting from $j$ to $i$ also incurs a fixed cost $c_f_{ij}$ to enter the market.

Firms are assumed to behave like monopolistic competitors: while there are only a finite number of firms, each is assumed to be small enough that it ignores the impact of its pricing decision on the behaviour of its competitors, and on national aggregate price indices. Facing CES demand, firms set a local price as a fixed mark-up over the domestic costs of production, determined by the elasticity of substitution, $\epsilon$

$$p_j(l) = \frac{\epsilon}{\epsilon - 1} c_j a(l)$$

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6The distribution could be generalised to include country-specific location parameters in the Fréchet distribution, as in Eaton & Kortum (2002), but since these would not be separately identifiable from other parameters of the model they are suppressed to minimise the notation.
and an export price that also embodies the marked-up unit transportation costs
\[ \tilde{p}_i(l) = \frac{\epsilon}{\epsilon - 1} \tau_{ij} c_j a(l) \]

This yields revenues from market \( i \) for the firm of
\[ \tilde{p}_i(l) x_i(l) = \frac{\tilde{p}_i(l)^{1-\epsilon} Y_i}{P_i^{1-\epsilon}} = \left( \frac{\epsilon}{\epsilon - 1} \frac{\tau_{ij} c_j a(l)}{P_i} \right)^{1-\epsilon} Y_i \]
and profits
\[ \Pi_{ij}(l) = \frac{1}{\epsilon} \left( \frac{\epsilon}{\epsilon - 1} \frac{\tau_{ij} c_j a(l)}{P_i} \right)^{1-\epsilon} Y_i - c_j f_{ij} \]

Rewriting sales revenue as
\[ \tilde{p}_i(l) x_i(l) = R_{ij}(l) = \beta_{ij} a(l)^{1-\epsilon} \]
where \( \beta_{ij} \equiv \left( \frac{\epsilon}{\epsilon-1} \frac{\tau_{ij} \epsilon}{P_i} \right)^{1-\epsilon} Y_i \) then a firm finds it profitable to enter an export market if and only if
\[ a^{1-\epsilon} \geq \frac{c_j f_{ij}}{\beta_{ij}} \]

Redefining \( F_{ij} \equiv \epsilon c_j f_{ij} \), equivalently entry occurs if and only if
\[ a \leq \left( \frac{F_{ij}}{\beta_{ij}} \right)^{\frac{1}{1-\epsilon}} \]

2.1 The Distribution of Firm Revenues

In the analysis that follows, the derived distributions should all be understood to be conditional on the realisations of national price indices and GDPs, and that individual firms are small enough that the realisation of their productivity draw is independent of aggregate national variables GDP and P. This conditionality is suppressed to simplify notation.

Under the assumption that unit input costs are distributed Fréchet, firm sales inherit a Weibull (\( \beta_{ij}, \frac{\epsilon}{\epsilon-1} \)) distribution.

\[ F \left( \beta_{ij} a(l)^{1-\epsilon} \leq x \right) = F \left( a(l) \geq \left( \frac{x}{\beta_{ij}} \right)^{\frac{1}{1-\epsilon}} \right) = 1 - e^{-\left( \frac{x}{\beta_{ij}} \right)^{\frac{1}{1-\epsilon}}} \]
This is not particularly tractable, but in the special case in which unit input requirements are distributed Fréchet \((\epsilon - 1)\) this collapses to an exponential distribution:

\[
F(\beta_{ij}a(l)^{1-\epsilon} \leq x) = 1 - e^{-\frac{x}{\beta_{ij}}}
\]

with a pdf for firm revenues

\[
f(R_{ij}(l) = x) = \frac{1}{\beta_{ij}} e^{-\frac{x}{\beta_{ij}}}
\]

The probability that firm \(l\) enters market \(i\) is

\[
F(\beta_{ij}a(l)^{1-\epsilon} \geq F_{ij}) = e^{-\frac{F_{ij}}{\beta_{ij}}}
\]

so the distribution of firm revenues, conditional on entry taking place is

\[
f(R_{ij}(l) = x|R_{ij}(l) > 0) = \frac{1}{\beta_{ij}} e^{-\frac{x-F_{ij}}{\beta_{ij}}}
\]

This conditional distribution, which models the intensive margin of firm sales that we observe in the data, is no longer exponential, as its mean has been shifted right by the selection into the market of only those firms whose sales can exceed the minimum cut-off sales \(F_{ij}\) necessary to generate profits sufficient to cover the fixed costs. While the distribution is no longer exponential, it retains the tractable aggregative properties of the exponential family. \(\beta_{ij}\) is the average sales of a firm \(l\) from \(j\) successfully entering market \(i\) in excess of the minimum threshold.

\[
E[U_{ij} = R_{ij} - F_{ij}|R_{ij} > 0] = \beta_{ij} \Rightarrow E[R_{ij}^l|R_{ij}^l > 0] = F_{ij} + \beta_{ij}
\]

### 2.2 The Distribution of Aggregate Bilateral Trade

Aggregate trade volumes reflect the cumulative sales of all the participating firms. Given a model of both the extensive and intensive margins (firm entry and firm sales), we can derive a distribution for aggregate trade volumes that is fully micro-founded in firms’ productivity shocks and estimable by standard maximum likelihood techniques, without introducing atheoretic statistical error terms.

Conditional on the number of firms entering, it is straightforward to derive the distribution of their aggregate trade. The convolution of the sum

\footnote{This is a very strong assumption, which may not hold in practice. Section 3.3 explores how robust estimates may be outside of this special case.}
of n i.i.d. firm-level exports yields a distribution for aggregate trade, \( X_{ij} \), conditional on the number of successfully exporting firms, \( n \). See Appendix A for a derivation.

\[
f(X_{ij} = x|n_{ij} = k) = \begin{cases} 
0 & \text{if } x < kF_{ij} \\
\frac{(x-kF_{ij})^{k-1}}{(k-1)! \beta_{ij}} e^{-\frac{1}{\beta_{ij}}(x-kF_{ij})} & \text{if } x \geq kF_{ij} 
\end{cases}
\]

The only source of randomness in the model is firms’ unit input requirement draw, \( a(l) \). Assuming that each of these draws is i.i.d, with \( N_j \) firms drawing productivities, and each firm having a probability of successfully entering market \( i \) of \( e^{-\frac{F_{ij}}{\pi_{ij}}} \), the number of firms entering market \( i \) from \( j \) is distributed Binomial \( (N_j, e^{-\frac{F_{ij}}{\pi_{ij}}}) \).

As is well known, for large \( N \) and small \( p \) the Binomial distribution is well approximated by a Poisson \( (Np) \). Making use of that approximation, the number of firms entering the market is distributed approximately Poisson \( (N_j e^{-\frac{F_{ij}}{\pi_{ij}}}) \).

The benefits of making the Poisson approximation of the Binomial extensive margin firm-entry process is that we can treat the number of domestically active firms \( N_j \), which is unobserved, as a parameter to be estimated by maximum likelihood. The Binomial distribution enforces the constraint that \( N_j \) is an integer, which makes the likelihood non-differentiable w.r.t \( N_j \), whereas the Poisson allows one to treat \( N_j \) as a continuous rather than discrete variable, facilitating estimation.

Given a distribution for the entry of firms and a distribution for aggregate trade conditional on the number of entrants, we can derive the unconditional distribution of aggregate trade.

\[
f(X_{ij} = x) = \sum_{k=0}^{\infty} f(X_{ij} = x, n = k) = \sum_{k=0}^{\infty} f(n = k) f(X_{ij} = x|n = k)
\]

Since \( f(X_{ij} = x|n = k) = 0 \) for \( x < kF_{ij} \), the minimum export level \( F_{ij} \) required by the fixed costs of market entry imply a maximum number of participating firms given total aggregate trade, \( x \), \( k_{max} = \text{floor}\left( \frac{x}{F_{ij}} \right) \).

\[
f(X_{ij} = x|n = k) > 0 \Rightarrow k < \frac{x}{F_{ij}}
\]
This ceiling on the number of exporting firms implies that integrating out the number of unobserved participating firms involves calculating a finite, rather than an infinite, sum, which makes precise estimation feasible without an approximation.

Denoting the Poisson probability of firm entry as $\lambda = N_j e^{-\frac{F_{ij}}{\beta_{ij}}}$

$$f(X_{ij} = x) = \begin{cases} 
    e^{-\lambda} & \text{if } x = 0 \\
    0 & \text{if } 0 < x < F_{ij} \\
    \sum_{k=1}^{\lfloor \frac{x}{F_{ij}} \rfloor} e^{-\lambda \beta_{ij} (x-kF_{ij})^{k-1}} \left( \frac{\lambda}{k!} \right)^k e^{-\frac{x}{\beta_{ij}}} (x-kF_{ij})^{k-1} \beta_{ij}^k & \text{if } x \geq F_{ij} 
\end{cases}$$

We can exploit the fact that similar fundamental forces drive the intensive and extensive margins to express $\lambda$ in terms of its fundamentals, $N_j$, $\beta_{ij}$ and $F_{ij}$, allowing for joint estimation of the impact of trade frictions $\beta_{ij}$ and $F_{ij}$ on both margins.

$$f(X_{ij} = x) = \begin{cases} 
    e^{-N_j e^{-\frac{F_{ij}}{\beta_{ij}}}} & \text{if } x = 0 \\
    0 & \text{if } 0 < x < F_{ij} \\
    \sum_{k=1}^{\lfloor \frac{x}{F_{ij}} \rfloor} e^{-N_j e^{-\frac{F_{ij}}{\beta_{ij}}}} \left( \frac{N_j}{\beta_{ij}} \right)^k \left( \frac{\beta_{ij}}{k!} \right)^k e^{-\frac{x}{\beta_{ij}}} (x-kF_{ij})^{k-1} & \text{if } x \geq F_{ij} 
\end{cases}$$

### 3 Estimation

The parameters of interest, $N_j$, $\beta_{ij}$ and $F_{ij}$ are identifiable due to the highly non-linear form of the likelihood, which is smooth and can be estimated by maximum likelihood.

$\beta_i$, the average firm sales in excess of the minimum, depends on the structural parameters of the model as follows

$$\beta_{ij} \equiv \left( \frac{\epsilon}{\epsilon - 1} \frac{\tau_{ij} c_j}{P_i} \right)^{1-\epsilon} Y_i$$

Both the ideal price index terms $P_i$ and GDP $Y_i$ are themselves endogenous functions of the model’s structural trade frictions. Rather than trying to solve for them as functions of the set of $\tau_{ij}$, $c_j$ and $f_{ij}$, they can be controlled for in an estimation by including importing country dummies to absorb them. Similarly, unobserved differences in unit costs $c_j$ across producers can be controlled for by including exporter fixed effects. This suggests estimating

$$\log \beta_{ij} = \beta_i + \beta_j + \bar{\beta}_r \tau_{ij} \quad (3)$$
where $\tilde{\tau}_{ij}$ is a vector of bilateral controls, such as distance, sharing a common border or colonial history, etc.

Similarly, $F_{ij} \equiv \epsilon c_j f_{ij}$ suggests estimating

$$\log F_{ij} = \phi_i + \phi_j + \tilde{\phi}_f f_{ij}$$  \hspace{1cm} (4)

where $f_{ij}$ is a vector of bilateral controls, such as regulations on market access, which might affect the fixed costs of market entry.

### 3.1 Simulating a Sample of Trade Flows

In order to validate the performance of the model, we simulate a sample of trade flows. Given the structural parameters of the model, $\kappa$, $N_j$, $\beta_{ij}$ and $F_{ij}$, for each exporting country, $j$, we generate $N_j$ firms by drawing $N_j$ unit labour requirements from a Fréchet ($\kappa$) distribution. For each potential export market, $i$, the productivity draw of each firm in $j$ is compared to the entry condition (2) determined by $\beta_{ij}$ and $F_{ij}$. If a firm’s labour productivity is high enough to enter market $i$, its sales are determined by (1). Aggregate exports from $j$ to $i$ are determined as the sum of the exports of successful firms in $j$. The model is then estimated on this simulated sample of synthetic trade flows.

The synthetic sample represents trade flows between 40 exporting countries and 40 potential markets. The number of firms taking a productivity draw in each market was drawn from a uniform distribution between 250 and 450. The parameter governing the distribution of firm productivity, $\kappa = 8.28^8$. This implies $\epsilon = 9.28$.

The determinants of average firm sales from $j$ to $i$ ($\beta_{ij}$) are modelled as in equation (3), as a combination of importer and exporter fixed effects, and bilateral variables. Since a full set of importer and exporter dummies are perfectly collinear, we normalise $\beta^X_1 = \beta^M_1$ for the first importer and exporter, where $e^{\beta^X_1}$ was drawn from a uniform distribution between 900000 and 925000. The fixed effect for exporter $j$ is given by $\beta^X_j$, and the fixed effect for importer $i$ is given by $\beta^M_i$, where $e^{\beta^X_j}$ and $e^{\beta^M_i}$ were drawn from a uniform distribution between 1 and 3. A bilateral trade barrier, $\beta_{ij}$, which we will label as ‘distance’, has pairwise variation, with $e^{\beta_{ij}}$ drawn from a uniform distribution between 2000 and 18000. $\tau_{\text{dist}} = -1$.

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8This value is taken from Eaton & Kortum (2002), which introduced the Fréchet distribution into the empirical trade literature, although as noted above the interpretation of the distribution differs between the two papers, so that this parameter in actual trade data might be quite different.
The determinants of fixed costs are modelled in a similar way, following equation (4). We normalise $F^X_i = F^M_i$, with $e^{F^X_i}$ drawn from a uniform distribution between 100 and 110. The country specific component of an exporter $j$’s fixed costs is given by $F^X_i + F^X_j$, the country specific component of importer $i$’s fixed costs is given by $F^M_i + F^M_j$, where $e^{F^M_i}$ and $e^{F^X_j}$ were drawn from a uniform distribution between 1 and 3. A bilateral trade barrier, $F_{ij}$, which we will label as ‘regulation’, has pairwise variation, with $e^{F_{ij}}$ drawn from a uniform distribution between 2000 and 18000. $\tau_{reg} = 0.2$.

Given this structure of trade barriers, the underlying firm productivity draws are made, yielding the aggregate trade flows. Of the 1600 potential bilateral trade flows, 752 saw some positive trade. The distribution of trade flows, and the number of firms entering export markets, are presented in Figures (1) to (4). Most markets are characterised by only a small number of firms entering, as only the most productive in each country can expect to generate high enough sales to offset the fixed costs of market entry. On average 29.7% of a country’s firms do some exporting, and this ranges between 2.1% for the most isolated, to 62.4% for the most integrated country.

Even within the set of positive bilateral trade flows, the majority are “small”, as most bilateral trade routes can only support a small number of firms. The distribution of aggregate bilateral trade flows closely mirrors the underlying distribution of exporting firms, which is a Weibull with the same $\kappa$ parameter inherited from the unit labour requirement process. $\kappa > 1$ will deliver a thin-tailed distribution of firms’ sales. A fat-tailed empirical distribution of trade flows will arise if $\kappa < 1$ and is also consistent with the underlying statistical model, and the structural trade barriers could still be estimated using the same maximum likelihood procedure on data with fat tails.

The estimated elasticities of the trade barriers with bilateral variation, distance and regulation, are reported in Table 1. They are very close to the true parameter values, and are estimated with small standard errors. Since the individual coefficients on the many fixed effects are of limited interest, their estimation is summarised in Figures (5)-(8), which present the relationship between the estimated (x-axis) and true (y-axis) parameter values, as well as 95% confidence intervals vertically for each coefficient estimate. For all five sets of variables, estimates lie very close to the true parameter values on the 45 degree line. The precision of the estimation varies by parameter types, with the greatest uncertainty associated with the number of exporting firms. The fixed costs $F^M_i$ and $F^X_j$ are estimated most precisely.

9 The standard error for $N_j$ has been calculated by the delta method, since the coefficient estimated in the model is $n_j = log(N_j)$, to constrain the estimated number of potential
Overall, the model seems to do a very good job of estimating the underlying structural parameters. This may not be too surprising, as the model’s assumptions are well aligned with the data generating process. They are not perfectly aligned, as the model does not reflect that the same productivity draws by firms in country $j$ will determine exports from $j$ to $i$ as well as from $j$ to $l$, so there may be additional correlation in exports from the same source country, $X_{ij}$ and $X_{lj}$. For example an unusually high productivity draw for a firm in $j$ will lead to higher exports in all of $j$’s markets, not just one. However, this problem is diminishing in the number of potential exporters, as the distribution of draws will converge to the same underlying distribution for all bilateral partners, reducing the influence of any individual outlying high productivity realisation, which might influence a small sample.

With estimates of the structural parameters in hand, we can also construct estimates of the number of firms that select into each export market. In section 2.2 we derived the distribution of aggregate bilateral trade, conditional on $n_{ij}$ firms exporting from $j$ to $i$. Given parameter estimates for $\hat{\beta}_{ij}$ and $\hat{F}_{ij}$, the conditional likelihood for aggregate trade only depends on $k$. We can evaluate the likelihood for all feasible $k < \frac{x_{ij}}{F_{ij}}$, leading to a maximum-likelihood estimate of the number of participating exporters, $\hat{k}$, as the number that maximises the conditional likelihood (given the estimates of the model’s structural parameters).

Figure (10) presents the $\hat{k}$ on each route versus the actual number of firms that selected into trade on each bilateral route in our sample with positive trade. Although this is not a parameter of the model, and so is not a variable that is estimated directly, the model is very successful at predicting it.

It may be surprising that the model is able to identify such a rich set of parameters just from observations of aggregate trade flows. Consideration of the likelihood function shows that all observations, whether they are positive or negative, contain information about all of the parameters ($N_j$, $\beta_{ij}$ and $F_{ij}$) enter the likelihood in both case.

The fixed cost parameters appear to be most accurately estimated, and there is some intuition for why this is likely to be the case. Since positive trade flows are a zero-probability event if $x_{ij} < F_{ij}$, each observation of positive exports places a constraint on the constituent elements of the bilateral fixed costs that acts as a ceiling, preventing them on average from rising too high. On the other hand, the presence of country-pairs with zero bilateral trade makes it unlikely that fixed costs are low, as otherwise even a small number of very unproductive firms would be able to achieve small positive exports. These two effects, from both positive and zero trade flows, help to identify firms to be non-negative.
the elements of the fixed costs very accurately, and they do not depend very strongly on the accuracy of the estimation of the model’s other key parameters.

Identification of $N$ and $\beta$ is more interdependent. Given a structure of fixed costs, average trade flows could represent a small number of exporters with very high average sales, or a large number of firms with small average sales. Identification may be achieved by the differential effects that an increase in $N$ and $\beta$ have on firms’ selection into markets. When more firms take productivity draws, more firms will get over the threshold to export, but the distribution of the productivity of successful exporters will be unchanged. However, when $\beta$ rises, not only does average productivity and hence sales of all existing exporters rise, but the threshold to export also falls, allowing a marginally less productive firm to now enter the export market. This changes the distribution of average productivity of the successfully exporting firms, dampening the rise in the average by diluting it with marginally less productive firms.

This is the same effect that Helpman et al. (2008) identify as the heterogeneity effect, and the structure of the model provides the linkage between aggregate trade flows and average firm sales that helps to identify the determinants of $\beta$. The maintained assumptions on the shape of these distributions allow the model to accurately determine whether the existing pattern of trade was generated by a relatively small number of high output firms or a large number of low output firms.

3.2 Comparing the Poisson-Gamma to Other Estimation Models

It is interesting to compare estimates from the Poisson-Gamma model to those from traditional techniques of gravity equation estimation. Traditional log-linearisation and OLS estimation faces several econometric issues, such as the sample selection, heteroskedasticity, and heterogeneity biases discussed in the literature. The Poisson model of Santos Silva & Tenreyro (2006) is intended to address the first two, while Helpman et al. (2008)’s model is intended to address the first and third. However, addressing all three simultaneously has proved difficult.

We take the same synthetic dataset and reestimate a gravity equation using OLS, Santos-Silva and Tenreyro’s Poisson estimator, and Helpman, Melitz and Rubinstein’s two-stage non-linear least squares estimator. The results of the key parameter estimates, $\tau_{\text{dist}}$ and $\tau_{\text{reg}}$ are presented in Table 2.

When comparing the results for fixed costs, it should be borne in mind that
the parameter $\tau_{\text{reg}}$ represents the contribution of regulation to fixed costs, and has a positive sign - more regulations raise bilateral entry barriers. However, in most empirical frameworks estimates will appear with a negative sign, reflecting the negative impact of higher fixed costs on trade flows. I report coefficients as they are estimated by each model. The disagreement across models is in the magnitude of estimates of $\tau_{\text{reg}}$, not its sign.

The elasticity estimates from a traditional log-linearised OLS gravity equation are badly biased up (in terms of absolute magnitudes). This is unsurprising, given the well-known biases that affect this specification.

We can try to control for sample selection by running a two-step Heckman sample selection correction. The inverse Mills ratio is statistically significant in the second stage, but does not dramatically affect the magnitude or biases in the coefficient estimates.

A Poisson specification is less biased than OLS. The improvement appears to come principally through the reduction in the heteroskedasticity bias, as comparing Poisson estimates on the full sample, including zeros, and the sample of positive trade flows, the estimate on the positive sample is more accurate. Santos Silva & Tenreyro (2006) also found that the coefficients in a Poisson estimation did not change much with the inclusion of zero trade flows, concluding that sample selection bias was less important than heteroskedasticity bias.

HMR’s methodology controls for the bias introduced by aggregating up over the exports of firms with heterogeneous productivity levels, which acts like an omitted variable correlated with the trade barriers due to the active selection of firms into export markets. They show that a two-step estimation procedure can control for this, estimating a Heckman sample selection procedure in a first-stage of whether any firms export (the “extensive” margin). This can then be used as a control for the average productivity of active firms when estimating the “intensive” margin of trade in the second stage.

However, on this sample, the method is not very successful. While both the sample selection and heterogeneity bias terms enter very statistically significantly in the intensive margin, the estimate on the distance variable actually moves further away from the true parameter in the intensive margin regression. The coefficient on the regulation variable in the extensive margin (estimated by probit), is also too large in absolute magnitude. The non-robustness of the HMR method to estimate the intensive and extensive margins may reflect its vulnerability to heteroskedasticity-bias, as emphasised by Santos Silva & Tenreyro (2014).

A detailed comparison between the results of the Poisson-Gamma and HMR models is interesting, because they are both derived from a very similar firm-level environment. This makes a comparison of the parameter estimates
more meaningful, as they are both intended to recover the same underlying objects, the determinants of firms’ average bilateral sales revenue, and bilateral fixed costs. Figures (11)-(14) provide further evidence that HMR’s methodology may not provide reliable estimates. These figures present estimates of importer and exporter fixed effects on the intensive and extensive margins, using HMR’s technique, and compares them to the true parameter values. The intensive margin estimates in Figures (11) and (12) are correlated with the true parameters, but the estimates appear to be negatively biased (the true parameters generally lie above the 45 degree line), and the standard errors appear inaccurate (many of the true values do not lie within the 95% confidence intervals). There is a similar pattern on the extensive margin, with distortions particularly strong for countries with low fixed costs, which HMR’s estimates tend to suggest even more favourable market access than is the case. These difficulties may reflect that the first-stage probit relies on a homoskedasticity assumption that is strongly violated on this sample.

3.3 Robustness of the Poisson-Gamma Model

3.3.1 Divergence of $\epsilon$ and $\kappa$

One important assumption in the derivation of the Poisson-Gamma likelihood function was the restriction on the relationship between the shape of the Frechét distribution of productivity draws and the elasticity of substitution, $\kappa = \epsilon - 1$. This is a very strong assumption, imposing an unlikely linkage between the supply and demand sides of the model. Given that this assumption may not hold in practice, it is important to explore how robust the model is to it. One way to do this is to reestimate the model on a sample of synthetic trade flows generated in an environment in which $\kappa \neq \epsilon - 1$. This can give us a sense of the sensitivity of the model’s bias to this assumption.

We redraw firms’ productivity and trade decisions, holding $\epsilon = 9.28$ and the other structural parameters unchanged, but reducing $\kappa$ by 2 to 6.28. This increases the dispersion of firms’ productivity draws, and might be expected to increase aggregate trade flows, by giving more firms a chance of a sufficiently good productivity draw to get over the fixed cost thresholds. The number of bilateral trade routes with positive trade rises from 752 in the baseline sample to 886 on the new sample. The distribution of bilateral trade flows looks very similar, and closely follows the distribution of the number of exporters. The average fraction of firms that export in the new sample is

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For ease of comparison with the model, I flip the sign on HMR’s estimates on the extensive margin, so that positive fixed effects are associated with countries with higher barriers to market entry.
27.9%, ranging between 0.35% in the most isolated exporter, to 69.6% in the most integrated exporter.

Figures (15)-(19) illustrate how parameter estimates are affected by relaxing the assumption. The most dramatic effect is on the estimate of the number of local firms, $N_j$, which Figure (15) shows are quite badly biased up. The standard errors rise in size, reflecting the greater imprecision of estimation, but do not fully reflect it, as the true parameter frequently lies outside of the estimated 95% confidence interval. The bias in $\hat{N}_j$ is not unexpected. With a fattening of the tail of high productivity firms, a higher ratio of potential firms are able to enter export markets that expected given the other parameters. The model compensates for the observed increase in trade volumes by assuming that the number of potential exporters is larger.

The impact on the estimates of the parameters of the intensive margin is also noticeable, but less severe. Comparing Figures (16)-(17) to Figures (6)-(7), there is more volatility in the estimates, and the true parameter occasionally lies outside of the 95% confidence interval, which suggests the standard errors may not be completely accurate. However, there is not an obvious source of bias in the parameter estimates.

The estimates of the parameters of the fixed cost appear unaffected by the relaxation of the assumption. In Figures (18)-(19) there is a tight relationship between the estimated and true parameters, and the confidence intervals remain very narrow.

This pattern of a loss of accuracy in the estimation of the intensive margin, but high accuracy on the extensive margin is also apparent in the estimates of the trade barriers with country-pair variation. Table 3 reports the elasticity estimates for distance and regulation. There is a fall towards zero in the estimate of the impact of distance on the intensive margin, although the bias remains severe than in alternative estimation methods. The estimate of the elasticity of regulation on the extensive margin remains very close to the true parameter (although given how tightly the standard error is estimated, the 95% confidence interval no longer contains the true parameter).

Figure (20) repeats the analysis of maximising the conditional likelihood function, given first-stage parameter estimates of $\hat{N}_j$, $\hat{\beta}_{ij}$ and $\hat{F}_{ij}$, in order to estimate the number of firms that actually enter bilateral export markets, and compares the estimate to the actual number in the synthetic data. There is an apparent downward bias in the estimate, that becomes more pronounced in the larger trade flows.
### 3.3.2 Fat-Tailed (Weibull) Distribution of Firm Trade

A new set of trade flows are generated by resetting $\kappa = 0.75$, and $\epsilon = 1.75$. With $\kappa$ less than one, the Weibull distribution that governs individual firms’ sales has thick tails. On this new sample, 755 of the 1600 potential bilateral trade routes see positive trade flows. The new distribution of positive trade flows is presented in Figure (21).

Figures (22)-(26) illustrate the estimates of the fixed effect parameters, while Table 3 presents the coefficient estimates on the bilateral trade barriers. As expected, the Poisson-Gamma model is robust to fat-tailed distributions, since these are consistent with the model’s assumptions. The estimates of the bilateral trade elasticities are as accurate as in the baseline case. Estimates of the number of firms, and the country-specific elements of market size and fixed costs are estimated as accurately as in the baseline case, as illustrated in Figures (22)-(26). The model is also very successful at recovering the actual number of firms that trade on each route. The predictions in Figure (27) lie very close to the actual number of firms trading in the data.

### 4 Conclusion

This paper has proposed a new statistical model of trade in which the empirical variation used to identify the model is microfounded in the theory, without the need for any additional ad hoc error terms. This approach allows the possibility of simultaneously addressing the leading biases in estimating the determinants of trade flows, which current methods can only tackle in a piecemeal way. Through the lens of the model, one can exploit aggregate data to draw inferences about the forces affecting firm level behaviour, and estimate unobservable variables such as the number of exporting firms, and the number active in the domestic industry.

The model performs very well on a sample that was generated in an environment consistent with its structural assumptions, and outperforms other traditional estimation methods which cannot simultaneously address multiple sources of bias. However, results may be sensitive to the underlying data generating process. Results are most accurate when $\kappa$ is close to $\epsilon - 1$, although the strength of the bias as these diverge is most pronounced on the estimates of the number of firms, with a modest loss of accuracy in estimation of the intensive margin, and very little deterioration in the estimates of the fixed costs.

Obtaining estimates on a real dataset are a work in progress, but an interesting dataset to analyse is monthly US customs data, which reports
the number of shipments into individual US ports. With estimates of $\hat{\beta}_{ij}$ and $\hat{F}_{ij}$ in hand, the conditional likelihood $f(X_{ij}|n=k)$ can be evaluated at all feasible values of $k$, which would suggest the modal likely number of actual exporters. Some external validation of the model could then be obtained by comparing this prediction to the actual number of shipments, which may be a reasonable proxy for the number of active firms in a high frequency monthly dataset.

A Convolution of the sum of i.i.d. firm-level trade flows

Denoting the random variable $X_{ij}^{k-1}$ as the sum of $k-1$ firms’ exports, we aim to derive the distribution of $k$ successfully exporting firms, $X_{ij}^{k} = X_{ij}^{k-1} + X_{ij}^{1}$.

Assume that the distribution of $k-1$ exporters’ aggregate trade takes the form

$$f(X_{ij}^{k-1} = x) = \begin{cases} 0 & \text{if } x < (k-1)F_{ij} \\ \frac{1}{(k-2)!\beta_{ij}}e^{-\frac{1}{\beta_{ij}}(x-(k-1)F_{ij})} & \text{if } x \geq (k-1)F_{ij} \end{cases}$$

This can be verified for the case of a single exporting firm, in which

$$f(X_{ij}^{1} = x) = \begin{cases} 0 & \text{if } x < F_{ij} \\ \frac{1}{\beta_{ij}}e^{-\frac{x-F_{ij}}{\beta_{ij}}} & \text{if } x \geq F_{ij} \end{cases}$$

$$f(X_{ij}^{k} = x_k, X_{ij}^{k-1} = x_{k-1}) = f(X_{ij}^{k-1} + X_{ij}^{1} = x_k, X_{ij}^{k-1} = x_{k-1}) = f(X_{ij}^{1} = x_k - x_{k-1}, X_{ij}^{k-1} = x_{k-1})$$

Since firms’ productivity draws are i.i.d., the joint distribution of $X_{ij}^{k-1}$ and $X_{ij}^{1}$ is the product of their marginal distributions

$$f(X_{ij}^{1} = x_k - x_{k-1}, X_{ij}^{k-1} = x_{k-1}) = \begin{cases} 0, & \text{if } x_{k-1} < (k-1)F_{ij} \text{ or } x_k < kF_{ij} \\ \frac{1}{(k-2)!\beta_{ij}}e^{-\frac{1}{\beta_{ij}}(x_k - kF_{ij})}, & \text{if } x_{k-1} \geq (k-1)F_{ij} \text{ and } x_k \geq kF_{ij} \end{cases}$$

We recover the marginal distribution of $X_{ij}^{k}$ by integrating out with respect to all the possible realisations of $X_{ij}^{k-1}$ (Since each successful exporter
must sell at least $F_{ij}$, feasible realisations for $x_{k-1}$ are $(k-1)F_{ij} \leq x_{k-1} \leq x_k - F_{ij}$). If $x_k \geq kF_{ij}$

$$f(X_{ij}^k = x_k) = \int_{-\infty}^{\infty} f(X_{ij}^k = x_k, X_{ij}^{k-1} = x_{k-1}) dx_{k-1}$$

$$= \int_{(k-1)F_{ij}}^{x_k - F_{ij}} \frac{(x_{k-1} - (k-1)F_{ij})^{k-2}}{(k-2)!} e^{-\frac{1}{\beta_{ij}}(x_{k-1} - kF_{ij})} dx_{k-1}$$

Otherwise if $x_k < kF_{ij}$, either $x_1 < F_{ij}$ or $x_{k-1} < (k-1)F_{ij}$ and $f(X_{ij}^1 = x_k - x_{k-1}, X_{ij}^{k-1} = x_{k-1}) = f(X_{ij}^k = x_k) = 0$.

Making the change of variable $\phi = x_{k-1} - (k-1)F_{ij}$, if $x_k \geq kF_{ij}$

$$f(X_{ij}^k = x_k) = \int_{0}^{x_k - kF_{ij}} \frac{\phi^{k-2}}{(k-2)!} \beta_{ij}^{k} e^{-\frac{1}{\beta_{ij}}(x_k - kF_{ij})} d\phi$$

$$= \left[ \left. \frac{\phi^{k-1}}{(k-1)!} e^{-\frac{1}{\beta_{ij}}(x_k - kF_{ij})} \right|_{0}^{x_k - kF_{ij}} \right]$$

$$= \frac{(x_k - kF_{ij})^{k-1}}{(k-1)!} \beta_{ij}^{k} e^{-\frac{1}{\beta_{ij}}(x_k - kF_{ij})}$$

This has the same form as for $n = k - 1$ and $n = 1$ firms, and so by induction holds for any arbitrary $n$.

**References**


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Table 1: Estimates of the bilateral trade barrier elasticities in the baseline model

![Figure 1: Distribution of All Bilateral Trade Flows](image)
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Table 2: Comparison of Elasticity Estimates Across Methods
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Table 3: Estimates of the bilateral trade barrier elasticities with alternative values of $\kappa$ and $\epsilon$

Figure 2: Distribution of Positive Bilateral Trade Flows
Figure 3: Distribution of Number of Exporting Firms from Country $j$ to $i$

Figure 4: Distribution of Number of Exporting Firms from Country $j$ to $i$
(if positive)
Figure 5: Estimated vs Actual Number of Potential Exporting Firms, $N_j$

Figure 6: Estimated vs Actual Importer Sales Fixed Effects, $\beta^M_i$
Figure 7: Estimated vs Actual Exporter Sales Fixed Effects, $\beta_j^X$

Figure 8: Estimated vs Actual Importer Fixed Costs Fixed Effects, $F_i^M$
Figure 9: Estimated vs Actual Exporter Fixed Costs Fixed Effects, $F_j^X$

Figure 10: Estimated vs Actual Exporting Firms On Each Trade Route with Positive Trade
Figure 11: HMR Estimated vs Actual Importer Sales, $\beta_M^i$

Figure 12: HMR Estimated vs Actual Exporter Sales, $\beta_X^j$
Figure 13: HMR Estimated vs Actual Importer Fixed Costs, $F^M_i$

Figure 14: HMR Estimated vs Actual Exporter Fixed Costs, $F^X_j$
Figure 15: Estimated vs Actual Number $N_j$, $\kappa = 6.28$, $\epsilon = 9.28$
Figure 16: Estimated vs Actual $\beta^M_1$, $\kappa = 6.28$, $\epsilon = 9.28$

Figure 17: Estimated vs Actual $\beta^X_2$, $\kappa = 6.28$, $\epsilon = 9.28$
Figure 18: Estimated vs Actual Importer $F_i^M$, $\kappa = 6.28$, $\epsilon = 9.28$

Figure 19: Estimated vs Actual Exporter $F_j^X$, $\kappa = 6.28$, $\epsilon = 9.28$
Figure 20: Estimated vs Actual Exporting Firms, $\kappa = 6.28, \epsilon = 9.28$

Figure 21: Distribution of Positive Bilateral Trade Flows with $\kappa = 0.75$
Figure 22: Estimated vs Actual Number $N_j$, $\kappa = 0.75$

Figure 23: Estimated vs Actual Importer Sales Effects $\beta^M_i$, $\kappa = 0.75$
Figure 24: Estimated vs Actual Exporter Sales Effects $\beta^X_j$, $\kappa = 0.75$

Figure 25: Estimated vs Actual Importer Fixed Costs $F^M_i$, $\kappa = 0.75$
Figure 26: Estimated vs Actual Exporter Fixed Costs, $F_j^X$, $\kappa = 0.75$

Figure 27: Estimated vs Actual Exporting Firms, $\kappa = 0.75$